Photon creation in a spherical oscillating cavity

Francisco D. Mazzitelli * and Ximena Orsi Millán †

Departamento de Física Juan José Giambiagi,
Facultad de Ciencias Exactas y Naturales, Universidad de Buenos Aires,
Ciudad Universitaria, Pabellón 1, 1428 Buenos Aires, Argentina

We study the photon creation inside a perfectly conducting, spherical oscillating cavity. The electromagnetic field inside the cavity is described by means of two scalar fields which satisfy Dirichlet and (generalized) Neumann boundary conditions. As a preliminary step, we analyze the dynamical Casimir effect for both scalar fields. We then consider the full electromagnetic case. The conservation of angular momentum of the electromagnetic field is also discussed, showing that photons inside the cavity are created in singlet states.

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I. INTRODUCTION

The dynamical Casimir effect consists in the generation of photons from the vacuum state of the electromagnetic field in the presence of time-dependent boundaries or time dependent media [1, 2]. From the theoretical point of view, it is widely accepted that the most favorable scenario to observe the phenomenon involves a periodic time dependence, to enhance photon creation by parametric resonance. Although no concrete experiment has yet been performed to confirm this non-stationary Casimir effect, an experimental verification is not out of reach, and there are several interesting proposals and ongoing experiments to observe it [3].

Since the pioneering work by Moore [4], the dynamical Casimir effect for time dependent boundaries has been studied for different fields and geometries: scalar fields in one dimensional cavities [5, 6], and in three dimensional rectangular [7, 8], and spherical [9] cavities. For the electromagnetic field, it was analyzed in three dimensional rectangular [10] and cylindrical cavities with time dependent length [11] and radius [12]. The spherically symmetric situation has been considered to study quantum radiation from a time dependent interface between two dielectric media [13].

In this paper we consider the quantum electromagnetic field inside a perfectly conducting, spherical cavity with a time dependent radius. Although a spherical conducting shell may not be appealing from an experimental point of view, it presents many interesting theoretical aspects. On the one hand, there is no classical electromagnetic radiation for a pulsating sphere, not even if charged. Thus it is interesting to check whether the quantum effect exists or not. Moreover, the angular momentum conservation implies that, if the effect exists, photons should be created in singlet states. On the other hand, it is also of interest to compare the rate of TE and TM photon creation.

The paper is organized as follows. In Section II we describe the classical electromagnetic field inside a spherical cavity with time dependent radius. The TE (TM) modes are described by a scalar field satisfying Dirichlet (generalized Neumann) boundary conditions. For this reason, it is of interest to analyze the case of quantum massless scalar fields satisfying both boundary conditions, which we do in Section III. We study in detail the resonant case in which the cavity oscillates at twice the frequency of some field mode. In Section IV we quantize the electromagnetic field inside the cavity, and compute the number of TE and TM created photons. In Section V we discuss the conservation of the angular momentum of the electromagnetic field inside the cavity. Section VI contains our main conclusions. We use natural units $\hbar = c = 1$.

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* fmazzi@df.uba.ar
† xorsi@df.uba.ar

II. CLASSICAL ELECTROMAGNETIC FIELD INSIDE A SPHERICAL CAVITY

We consider a cavity bounded by a perfectly reflecting spherical shell. We will assume that the shell is at rest for $t < 0$, and that it moves following a given trajectory $a(t)$, for $0 < t < t_f$. The trajectory is prescribed for the problem and works as a time-dependent boundary condition for the field. Moreover, we will assume a non relativistic motion of the shell with $a(t) = a_0(1 + \epsilon f(t))$ with $\epsilon \ll 1$ and $f(t)$ a smooth function that vanishes for $t < 0$ and $t > t_f$.

The electromagnetic field inside the cavity can be described in terms of the four vector potential $A_\mu = (\varphi, \mathbf{A})$. In the Coulomb gauge, $\nabla \cdot \mathbf{A} = 0$, the scalar potential φ vanishes and the vector potential satisfies the wave equation $\Box \mathbf{A} = 0$.

In order to obtain the vector potential $\mathbf{A}$ we consider a function ϕ satisfying the scalar equation $\Box \phi = 0$. A solution of the wave equation for a given orientation of the axes must be solution for any other orientation, so we can obtain $\mathbf{A}$ starting from ϕ by means of $\mathbf{A} = \imath \mathbf{L} \phi = \mathbf{r} \times \nabla \phi$. This is not the more general solution for the electromagnetic field inside the cavity. For example, the electric field obtained from $\mathbf{A}$ has no component along $\hat{\mathbf{r}}$ (it is a TE mode). A linearly independent solution can be obtained, however, by interchanging the roles of $\mathbf{E}$ and $\mathbf{B}$. Therefore, the electromagnetic field can be written in terms of two vector potentials $\mathbf{A}^{TE}$ and $\mathbf{A}^{TM}$. Both of the them are obtained by the application of $\imath \mathbf{L}$ on solutions of the scalar wave equation, ϕ^{TE} and ϕ^{TM}, and satisfy the Coulomb gauge. The complete fields are

$$\mathbf{E} = \mathbf{E}^{TE} + \mathbf{E}^{TM} = -\partial_t \mathbf{A}^{TE} + \vec{\nabla} \times \mathbf{A}^{TM} \tag{1}$$

$$\mathbf{B} = \mathbf{B}^{TE} + \mathbf{B}^{TM} = \vec{\nabla} \times \mathbf{A}^{TE} + \partial_t \mathbf{A}^{TM} \tag{2}$$

The boundary conditions for the electromagnetic field on a moving interface between two media are [14]

$$\begin{aligned} (\mathbf{D}_{II} - \mathbf{D}_I).\hat{\mathbf{n}} &= \sigma \\ (\mathbf{B}_{II} - \mathbf{B}_I).\hat{\mathbf{n}} &= 0 \\ \{\hat{\mathbf{n}} \times (\mathbf{H}_{II} - \mathbf{H}_I) + (\mathbf{v}.\hat{\mathbf{n}})(\mathbf{D}_{II} - \mathbf{D}_I)\}.\hat{\mathbf{t}} &= \mathbf{K}.\hat{\mathbf{t}} \\ \{\hat{\mathbf{n}} \times (\mathbf{E}_{II} - \mathbf{E}_I) - (\mathbf{v}.\hat{\mathbf{n}})(\mathbf{B}_{II} - \mathbf{B}_I)\}.\hat{\mathbf{t}} &= 0 \end{aligned} \tag{3}$$

where $\hat{\mathbf{n}}$ denotes the normal to the interface going from medium I to medium II, $\hat{\mathbf{t}}$ is any unit vector tangential to the surface, σ is the surface charge density and $\mathbf{K}$ the surface current. These conditions can be derived using the Maxwell equations in the laboratory frame or, alternatively, by performing a Lorentz transformation in which a given part of the interface is instantaneously at rest, and imposing there the static boundary conditions [15].

We assume the spherical shell to be a perfect conductor, so the fields vanish in region II and the boundary conditions become

$$\mathbf{B}.\hat{\mathbf{r}} = 0 \qquad \{\mathbf{E} \times \hat{\mathbf{r}} + \dot{a}(t)\mathbf{B}\}.\hat{\mathbf{t}} = 0 \tag{4}$$

For the static cavity, we have $\dot{a}(t) = 0$ and the boundary conditions are the usual ones. In terms of the scalar fields these conditions read

$$\phi^{TE}|_{r=a_0} = 0 \qquad [\partial_r (r\phi^{TM})]|_{r=a_0} = 0 \tag{5}$$

When the shell begins to move Eq. (4) implies

$$\phi^{TE}|_{r=a(t)} = 0 \qquad \{(\partial_r + \dot{a}(t)\partial_t) r\phi^{TM}\}|_{r=a(t)} = 0 \tag{6}$$

From this discussion we see that the behavior of the vector potential $\mathbf{A}^{TE}$ ($\mathbf{A}^{TM}$)is related to the problem of a scalar field subjected to Dirichlet (generalized Neumann) boundary conditions. The description in terms of independent TE and TM fields is possible due to the particular geometry we are considering. Indeed, using the above definitions and boundary conditions it is easy to check that no mixed terms appear in Maxwell's Hamiltonian.

The two scalar functions ϕ^{TE} and ϕ^{TM} are known as Debye potentials [16]. These functions are related with the Hertz potentials Π_e and Π_m used in Ref. [11] by

$$\Pi_e = \phi^{TM}\mathbf{r} \qquad\qquad \Pi_m = \phi^{TE}\mathbf{r} \tag{7}$$

For the sake of simplicity, instead of dealing with the full electromagnetic case, we will first study the problem of scalar fields.

III. QUANTUM SCALAR FIELDS INSIDE A SPHERICAL SHELL

Let us consider a massless scalar field $\phi(\mathbf{r},t)$ inside a spherical cavity described by the Lagrangian $\mathcal{L}$

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi \tag{8}$$

The field operator can be written in terms of creation and annihilation operators as

$$\phi(\mathbf{r},t) = \sum_{k\ell m} a^{in}_{k\ell m}\varphi_{k\ell m}(\mathbf{r},t) + h.c. \tag{9}$$

where the mode functions $\varphi_{k\ell m}(\mathbf{r},t)$ form a complete orthonormal set of solutions of the wave equation.

A. Static cavity

When $t < 0$ we have a static spherical shell of radius a_0 and the field modes are given by

$$\varphi_{k\ell m}(\mathbf{r},t) = \phi_{k\ell m}(\mathbf{r})\frac{e^{-\imath\omega_{\ell k}t}}{\sqrt{2\omega_{\ell k}}} \tag{10}$$

where

$$\phi_{k\ell m}(\mathbf{r}) = \mathcal{C}_{k\ell m}\, \jmath_\ell(\omega_{\ell k}r)Y_{\ell m}(\theta,\phi) \tag{11}$$

are the eigenfunctions of the Laplacian with eigenvalues $-\omega^2_{\ell k}$, j_ℓ are the spherical Bessel functions and $Y_{\ell m}$ the spherical harmonics. The normalization constants $\mathcal{C}_{k\ell m}$ depend on the boundary conditions satisfied by the field.

We will consider two scalar fields, one satisfying Dirichlet boundary conditions and the other one satisfying generalized Neumann boundary conditions. It is worth to note that the scalar fields considered here are not exactly the Debye potentials of the electromagnetic field (the Lagrangian in Eq.(8) is not the Maxwell Lagrangian expressed in terms of the Debye potentials). In spite of this, we will denote them by ϕ^{TE} and ϕ^{TM} to stress the boundary conditions that they satisfy.

For the scalar field ϕ^{TE} the normalization constant is given by

$$C_{k\ell m} = \sqrt{\frac{2}{a_0^3}}\frac{1}{\jmath'_\ell(\jmath_{\ell k})} \tag{12}$$

with $\jmath_{\ell k}$ the k-th zero for the spherical Bessel function $\jmath_\ell(x)$. The frequencies of the modes are $\omega_{\ell k} = \frac{\jmath_{\ell k}}{a_0}$

On the other hand, for the scalar field ϕ^{TM} we have

$$C_{k\ell m} = \sqrt{\frac{2}{a_0^3}} \frac{1}{\jmath'_\ell(\kappa_{\ell k})} \frac{1}{\sqrt{\kappa_{\ell k}^2 - \ell(\ell+1)}} \tag{13}$$

where $\kappa_{\ell k}$ is the k-th zero of $\{\partial_x[x\jmath_\ell(x)]\} = 0$. The TM frequencies are given by $\omega_{\ell k} = \frac{\kappa_{\ell k}}{a_0}$.

The operators $a^{\dagger in}_{k\ell m}$ and $a^{in}_{k\ell m}$ create and annihilate particles with well defined energy, total and z-component of the angular momentum ℓ and m respectively. They correspond to the particle notion in the "in" region ($t < 0$).

B. Moving cavity

1. Dirichlet Boundary Condition

When the radius of the cavity depends on time, the modes of ϕ^{TE} can be expanded in terms of an instantaneous basis

$$\varphi^{TE}_{k\ell m}(\mathbf{r}, t) = \sum_p Q^{(k)}_{p,TE} \phi^{TE}_{p\ell m}(\mathbf{r}, a(t)) \tag{14}$$

with

$$\phi^{TE}_{p\ell m}(\mathbf{r}, a(t)) = \sqrt{\frac{2}{a^3(t)}} \frac{1}{\jmath'_\ell(\jmath_{\ell p})} \jmath_\ell(\frac{\jmath_{\ell p} r}{a(t)}) Y_{\ell m}(\theta, \phi) \tag{15}$$

Because of the spherical symmetry, in the expansion of the mode $\varphi^{TE}_{k\ell m}(\mathbf{r}, t)$ it is enough to use the functions $\phi^{TE}_{p\ell m}$ with the same values of ℓ and m, i.e. we only mix the first quantum number. Although the coefficients $Q^{(k)}_{p,TE}$ depend on the angular momentum ℓ (see Eqs.(16-17) below), in order to keep the notation as simple as possible we do not write this explicitly.

The initial conditions for the coefficients $Q^{(k)}_{p,TE}(t)$ are

$$Q^{(k)}_{p,TE}(t=0) = \frac{\delta_{kp}}{\sqrt{2\omega_{\ell k}}} \qquad \dot{Q}^{(k)}_{p,TE}(t=0) = -\imath \sqrt{\frac{\omega_{\ell k}}{2}} \delta_{kp} \tag{16}$$

These conditions ensure that, as long as $a(t)$ and $\dot{a}(t)$ are continuous at $t = 0$, each field mode and its time derivative are also continuous functions.

The expansion in Eq.(14) for the field modes must be a solution of the wave equation. Taking into account that at each time the functions $\phi^{TE}_{p\ell m}(\mathbf{r}, a(t))$ form a complete and orthonormal set, the wave equation is equivalent to the following set of coupled equations for $Q^{(k)}_{n,TE}(t)$

$$\begin{aligned} \ddot{Q}^{(k)}_{n,TE}(t) + [\omega_{\ell n}(t)]^2 Q^{(k)}_{n,TE}(t) &= -2\lambda(t) \sum_p \dot{Q}^{(k)}_{p,TE}(t) g^\ell_{pn} \\ &- \dot{\lambda}(t) \sum_p Q^{(k)}_{p,TE}(t) g^\ell_{pn} + \mathcal{O}(\epsilon^2) \end{aligned} \tag{17}$$

where $\lambda(t) = \frac{\dot{a}(t)}{a(t)}$ and

$$g^\ell_{pn} = a(t) \int_0^{2\pi} \int_0^\pi \int_0^{a(t)} d^3x \frac{\partial \phi^{TE}_{p\ell m}}{\partial a(t)} \phi^{TE*}_{n\ell m} \tag{18}$$

The coefficients g^{ℓ}_{pn} can be computed explicitly using that, for $k \neq k'$,

$$\int_0^a r^2 \jmath_\ell(kr)\jmath_\ell(k'r)dr = \frac{a^2}{k'^2 - k^2}\{k\jmath_\ell(k'a)\jmath'_\ell(ka) - k'\jmath_\ell(ka)\jmath'_\ell(k'a)\} \tag{19}$$

The result is

$$g^{\ell}_{pn} = -g^{\ell}_{np} = \begin{cases} 0 & \text{if p = n} \\ \frac{2\jmath_{\ell n}\jmath_{\ell p}}{[\jmath_{\ell p}]^2 - [\jmath_{\ell n}]^2} & \text{if p} \neq \text{n.} \end{cases} \tag{20}$$

2. Neumann Boundary Condition

We now consider a scalar field satisfying the generalized Neumann boundary condition on the surface of the shell. To satisfy the boundary conditions for $t > 0$ we will expand the mode functions with respect to an instantaneous basis, as we did for the case of Dirichlet boundary condition. However, in this case the choice for the instantaneous basis is not so easy as before, because the boundary condition on the moving shell given in Eq.(6) involves a time derivative of the field.

The instantaneous basis can be obtained by means of a change of variables in the (t, r) plane [10], provided that in the new variables (η, ξ) the boundary condition is the usual one (i.e. no time derivative of the field)

$$\{\partial_\xi[\xi\phi^{TM}(\eta, \xi, \theta, \varphi)]\}|_{\xi = l(\eta)} = 0 \tag{21}$$

where $l(\eta)$ is the value of the coordinate ξ on the moving spherical mirror.

We define the line $\eta = const$ to be a slight modification of the line $t = const$, in such a way that it is orthogonal to the worldline of the mirror $(t, a(t)\hat{\mathbf{r}})$ at $r = a(t)$. The coordinate ξ is defined as the distance from $r = 0$ to r on the line $\eta = const$. Explicitly

$$\begin{aligned} \eta &= t + g(r, t) \\ \xi &= \int_0^r dr' \sqrt{1 + \frac{(\partial_{r'} g(r', t))^2}{[1 + \partial_t g(r', t)]^2}} \end{aligned} \tag{22}$$

where $g(r, t) = \mathcal{O}(\epsilon)$ and therefore $\xi = r + \mathcal{O}(\epsilon^2)$ and $l(\eta) = a(t) + \mathcal{O}(\epsilon^2)$.

In order to ensure the orthogonality between the line $\eta = const$ and the world line of the mirror, we impose [17]

$$g(a(t), t) = 0 \qquad \partial_r g(r, t)|_{r = a(t)} = -\dot{a}(t) \tag{23}$$

The function $g(r, t)$ is of course not unique. It can be expressed as $g(r, t) = \dot{a}(t)a(t)v(r/a(t))$, where $v(1) = 0$ and $v'(1) = -1$ (the prime denotes derivation with respect to the argument). There are many solutions to these conditions, implying a freedom for selecting the instantaneous basis. However, physical quantities like the number of created particles, the energy density inside the cavity and the angular momentum of the field are independent of the particular choice of $g(r, t)$ [10].

In the new coordinates, the instantaneous basis is the trivial one

$$\varphi^{TM}_{k\ell m}(\eta, \xi, \theta, \varphi) = \sum_p \sqrt{\frac{2}{l^3(\eta)}} \frac{1}{\jmath'_\ell(\kappa_{\ell p})} \frac{1}{\sqrt{\kappa^2_{\ell p} - \ell(\ell + 1)}} Q^{(k)}_{p,TM}(\eta)\jmath_\ell(\frac{\kappa_{\ell p}}{l(\eta)}\xi)Y_{\ell m}(\theta, \phi) \tag{24}$$

Returning to the (t, r) variables, each field mode can be expanded as follows

$$\varphi^{TM}_{k\ell m}(t, r, \theta, \varphi) = \sum_p [Q^{(k)}_{p,TM}(t) + \dot{Q}^{(k)}_{p,TM}(t)g(r, t)]\phi^{TM}_{p\ell m}(\mathbf{r}, a(t)) + \mathcal{O}(\epsilon^2) \tag{25}$$

with

$$\phi^{TM}_{p\ell m}(r,a(t)) = \sqrt{\frac{2}{a^3(t)}}\frac{1}{\jmath'_\ell(\kappa_{\ell p})}\frac{1}{\sqrt{\kappa^2_{\ell p}-\ell(\ell+1)}}\jmath_\ell(\frac{\kappa_{\ell p}}{a(t)}r)Y_{\ell m}(\theta,\phi) \tag{26}$$

As in the Dirichlet case, the coefficients $Q^{(k)}_{p,TM}$ depend on the number ℓ, but we do not write the dependence explicitly. Assuming that $a(t)$ and $\dot{a}(t)$ are continuous at $t=0$, and that the initial acceleration satisfies $\ddot{a}(0)=\mathcal{O}(\epsilon^2)$, the initial conditions for $Q^{(k)}_{p,TM}(t)$ are the same as those for $Q^{(k)}_{p,TE}(t)$, Eq.(16). The equation of motion for $Q^{(k)}_{n,TM}(t)$ is

$$\begin{aligned}\ddot{Q}^{(k)}_{n,TM}(t) + [\omega_{\ell n}(t)]^2 Q^{(k)}_{n,TM}(t) &= -2\lambda(t)\textstyle\sum_p \dot{Q}^{(k)}_{p,TM} g^\ell_{pn} - \dot{\lambda}(t)\sum_p Q^{(k)}_{p,TM} g^\ell_{pn} \\ &-2a^2(t)\dot{\lambda}(t)\textstyle\sum_p \ddot{Q}^{(k)}_{p,TM} s^\ell_{pn} - \sum_p \dot{Q}^{(k)}_{p,TM}[s^\ell_{pn}\ddot{\lambda}(t)a^2(t) - \lambda(t)\eta^\ell_{pn}] \\ &-\lambda(t)a^2(t)\textstyle\sum_p \partial^3_t Q^{(k)}_{p,TM} s^\ell_{pn} + \mathcal{O}(\epsilon^2)\end{aligned} \tag{27}$$

where the coefficients s^ℓ_{pn} , η^ℓ_{pn} and g^ℓ_{pn} are given by

$$s^\ell_{pn} = \int_0^{2\pi}\int_0^{\pi}\int_0^{a(t)} d^3x v \phi^{TM}_{p\ell m}(\mathbf{r},a(t))\phi^{TM*}_{n\ell m}(\mathbf{r},a(t)) \tag{28}$$

$$\begin{aligned}\eta^\ell_{pn} = \textstyle\int_0^{2\pi}\int_0^{\pi}\int_0^{a(t)} d^3x a^2(t)\{&[\partial^2_{rr}v - (\omega_{\ell p})^2 v]\phi^{TM}_{p\ell m}(\mathbf{r},a(t))\phi^{TM*}_{n\ell m}(\mathbf{r},a(t)) \\ &+\tfrac{2}{r}\partial_r v \partial_r[r\phi^{TM}_{p\ell m}(\mathbf{r},a(t))]\phi^{TM*}_{n\ell m}(\mathbf{r},a(t))\}\end{aligned} \tag{29}$$

$$g^\ell_{pn} = \begin{cases} \frac{\kappa^2_{\ell p}}{\kappa^2_{\ell p}-l(l+1)} & \text{if p =n} \\ \frac{\kappa_{\ell n}\kappa_{\ell p}}{(\kappa^2_{\ell n}-\kappa^2_{\ell p})}\sqrt{\frac{\kappa^2_{\ell p}-l(l+1)}{\kappa^2_{\ell n}-l(l+1)}} & \text{if p}\neq\text{ n.}\end{cases} \tag{30}$$

C. Creation of particles

We are interested in the number of particles created inside the cavity, so it is natural to look for harmonic oscillations of the shell that could enhance that number by means of resonance effects for some specific external frequencies Ω. So we study the following trajectory

$$a(t) = a_0(1+\epsilon\sin(\Omega t)) \tag{31}$$

Let us first consider the Dirichlet scalar field ϕ^{TE}. When $t>t_f$, (”*out*” region), the radius returns to its initial value a_0, the right hand side in Eq.(17) vanishes and the solution is

$$Q^{(k)}_{n,TE}(t>t_f) = A^{(k)}_{\ell n,TE}e^{\imath\omega_{\ell n}t} + B^{(k)}_{\ell n,TE}e^{-\imath\omega_{\ell n}t} \tag{32}$$

where $A^{(k)}_{\ell n,TE}$ and $B^{(k)}_{\ell n,TE}$ are constant coefficients to be determined by the continuity conditions at $t=t_f$. In these coefficients we write the dependence on ℓ explicitly.

For $t > t_f$ we can define a new set of operators $a^{out}_{k\ell m}$ and $a^{\dagger out}_{k\ell m}$, associated with the particle notion in the "out" region . The "in" and "out" operators are connected by means of the Bogoliubov transformation

$$a^{out}_{n\ell m} = \sum_k [B^{(k)}_{\ell n,TE} a^{in}_{k\ell m} + (-1)^m A^{(k)*}_{\ell n,TE} a^{in\dagger}_{k\ell -m}]\sqrt{2\omega_{\ell n}} \tag{33}$$

The number of "out" particles in the mode (n,ℓ,m) is given by

$$< \mathcal{N}_{n\ell m} >=< 0_{in}|a^{out\dagger}_{n\ell m} a^{out}_{n\ell m}|0_{in} >= 2\omega_{\ell n} \sum_k |A^{(k)}_{\ell n,TE}|^2 \tag{34}$$

To obtain the coefficients $A^{(k)}_{\ell n,TE}$ and $B^{(k)}_{\ell n,TE}$ we must solve Eq.(17) for the coefficients $Q^{(k)}_{n,TE}(t)$. They are of the same form as those that describe the modes of a scalar field satisfying Dirichlet boundary conditions in a three-dimensional rectangular cavity [7], and can be solved using Multiple Scale Analysis (MSA), see for example, Refs.[7, 18]. We will review here the main results and include the details in the Appendix.

The solution for the coefficients $A^{(k)}_{\ell n,TE}$ and $B^{(k)}_{\ell n,TE}$ depends on the relation between the external frequency Ω and the natural frequencies of the field in the cavity. There is parametric resonance when the external frequency Ω equals the sum of two eigenfrequencies of the cavity with the same angular momentum $\Omega = \omega_{\ell n} + \omega_{\ell p}$. For simplicity, in what follows we will analyze the particular case $\Omega = 2\omega_{\ell n}$. Using MSA, one can show that the modes (ℓ, n) and (ℓ, q) are coupled if any of the following conditions is satisfied

$$\Omega = \omega_{\ell n} - \omega_{\ell q} \qquad \Omega = -\omega_{\ell n} + \omega_{\ell q} \tag{35}$$

For Φ^{TE} the frequencies are given by the zeros of the spherical Bessel functions ($\omega_{\ell n} = \frac{\jmath_{\ell n}}{a_0}$) and so, the spectrum is qualitatively different depending on the value of the number ℓ. For $\ell = 0$ the spectrum is equidistant. In the particular case $\Omega = 2\omega_{0n}$, the set of equations for the coefficients $A^{(k)}_{0n,TE}$ and $B^{(k)}_{0n,TE}$ corresponds to that of a one-dimensional cavity excited with twice the lowest eigenfrequency. The equations are coupled, the number of particles grows linearly in time, and the energy increases exponentially [6].

On the other hand, for $\ell \neq 0$ the spectrum is not equidistant [19], and one can check that there is no coupling between modes. Eq.(17) reduces to the Mathieu equation for the modes with frequency $\omega_{\ell n}$. The Bogoliubov coefficients $A^{(k)}_{\ell n,TE}$ and $B^{(k)}_{\ell n,TE}$ and the number of created particles grows exponentially.

For the case of the scalar field ϕ^{TM} the situation is similar. The set of Eqs.(27) has the same form as the set that describes the modes of a scalar field in a three-dimensional rectangular cavity satisfying generalized Neumann boundary conditions [10], and can be solved again using MSA. As before, the spectrum is equidistant for $\ell = 0$ and non-equidistant for $\ell \neq 0$.

In the resonant case $\Omega = 2\omega_{LN}$ (L $\neq 0$), for both TE and TM modes the number of created particles is given by (see Appendix)

$$< \mathcal{N}_{NLm} >=< 0_{in}|a^{out\dagger}_{NLm} a^{out}_{NLm}|0_{in} >= \sinh^2(\gamma\epsilon t_f) \tag{36}$$

and

$$< \mathcal{N}_{NL} >= \sum_m < 0_{in}|a^{out\dagger}_{NLm} a^{out}_{NLm}|0_{in} >= (2L+1)\sinh^2(\gamma\epsilon t_f) \tag{37}$$

The constant γ determines the rate of growth. For TE modes we have

$$\gamma^{TE} = \frac{\jmath_{LN}}{2a_0} \tag{38}$$

while for TM modes

$$\gamma^{TM} = \frac{\kappa_{LN}}{2a_0} \frac{1 + \frac{L(L+1)}{\kappa^2_{LN}}}{1 - \frac{L(L+1)}{\kappa^2_{LN}}} \tag{39}$$

The case $\ell = 0$ is qualitatively different and, as stated above, equivalent to the one dimensional dynamical Casimir effect. However, as we will see in the next section, these modes are absent for the electromagnetic field.

IV. THE ELECTROMAGNETIC FIELD

In Section II we showed that the electromagnetic field inside the spherical cavity can be described in terms of two vector potentials. They can be expanded in terms of creation and annihilation operators as

$$\mathbf{A}^{TE}(\mathbf{r},t) = \sum_{k\ell m} a^{in}_{k\ell m}\mathbf{A}^{TE}_{k\ell m}(\mathbf{r},t) + h.c. \tag{40}$$

$$\mathbf{A}^{TM}(\mathbf{r},t) = \sum_{k\ell m} a^{in}_{k\ell m}\mathbf{A}^{TM}_{k\ell m}(\mathbf{r},t) + h.c. \tag{41}$$

The modes $\mathbf{A}^{TE}_{k\ell m}(\mathbf{r},t)$ and $\mathbf{A}^{TM}_{k\ell m}(\mathbf{r},t)$ can be obtained from the modes of two scalars fields through the application of the operator $\imath\mathbf{L} = \mathbf{r}\times\vec{\nabla}$. This operator acts only on the angular part of the scalar modes, so we introduce the vectorial spherical harmonics $\mathbf{X}_{\ell m}(\theta,\phi)$

$$\mathbf{X}_{\ell m}(\theta,\phi) = \frac{\mathbf{L}}{\sqrt{\ell(\ell+1)}}Y_{\ell m}(\theta,\phi) \tag{42}$$

(the additional factor $\sqrt{\ell(\ell+1)}$ is needed for normalization). Therefore the modes for the vector potentials are given by

$$\begin{aligned}\mathbf{A}^{TE}_{k\ell m}(\mathbf{r},t<0) &= \sqrt{\tfrac{2}{a_0^3}}\tfrac{1}{\jmath'_\ell(\jmath_{\ell k})}\jmath_\ell(\tfrac{\jmath_{\ell k}r}{a_0})\mathbf{X}_{\ell m}(\theta,\phi)\\ \mathbf{A}^{TM}_{k\ell m}(\mathbf{r},t<0) &= \sqrt{\tfrac{2}{a_0^3}}\tfrac{1}{\jmath_\ell(\kappa_{\ell k})}\tfrac{1}{\sqrt{\kappa^2_{\ell k}-\ell(\ell+1)}}\jmath_\ell(\tfrac{\kappa_{\ell k}r}{a_0})\mathbf{X}_{\ell m}(\theta,\phi)\end{aligned} \tag{43}$$

for the static cavity and

$$\begin{aligned}\mathbf{A}^{TE}_{k\ell m}(\mathbf{r},t>0) &= \textstyle\sum_p \sqrt{\tfrac{2}{a^3(t)}}\tfrac{1}{\jmath'_\ell(\jmath_{\ell p})}Q^{(k)}_{p,TE}(t)\jmath_\ell(\tfrac{\jmath_{\ell p}r}{a(t)})\mathbf{X}_{\ell m}(\theta,\phi)\\ \mathbf{A}^{TM}_{k\ell m}(\mathbf{r},t>0) &= \textstyle\sum_p \sqrt{\tfrac{2}{a^3(t)}}\tfrac{1}{\jmath'_\ell(\kappa_{\ell p})}\tfrac{1}{\sqrt{\kappa^2_{\ell p}-\ell(\ell+1)}}[Q^{(k)}_{p,TM}(t)+g(\mathbf{r},t)\dot{Q}^{(k)}_{p,TM}(t)]\jmath_\ell(\tfrac{\kappa_{\ell p}r}{a(t)})\mathbf{X}_{\ell m}(\theta,\phi)\end{aligned} \tag{44}$$

for the moving cavity. It is worth to remark that, as $\mathbf{L}Y_{00}=0$, there is no monopolar term in the expansions for the electromagnetic field.

The dynamical evolution of the TE (TM) modes is that of the modes of $\phi^{TE}(\phi^{TM})$ with $\ell\neq 0$. As a consequence, the number of created photons in each mode equals the number of created particles of the corresponding scalar field. If we consider again the parametric resonant case $\Omega = 2\omega_{LN}$, the number of photons grows exponentially and is given by the Eqs.(36,38) in the case of TE modes and by Eqs. (36,39) in the case of TM modes. It is interesting to note that it is in general not possible to excite at the same time both a TE and a TM mode (for that to be possible one should have $\omega^{TE}_{\ell n}+\omega^{TE}_{\ell n'} = \omega^{TM}_{\ell' k}+\omega^{TM}_{\ell' k'}$, which is not satisfied).

V. ANGULAR MOMENTUM

In this section we discuss the conservation of the angular momentum of the electromagnetic field inside the spherical cavity, showing that photons are created in singlet states. As we start in the vacuum state of the electromagnetic field, the average value of the angular momentum is initially zero. The oscillations of the radius of the cavity does not break the symmetry under rotations, so the angular momentum must vanish at all times.

The angular momentum of the electromagnetic field is

$$Ł = \int_{sphere} \mathbf{r} \times (\mathbf{E} \times \mathbf{B}) d^3x = Ł^{TE} + Ł^{TM} \tag{45}$$

As in resonant situations it is possible to produce either TE or TM photons, in what follows we will consider only one of the two polarizations, without specifying which one. Both $Ł^{TE}$ and $Ł^{TM}$ are of the form

$$\begin{aligned} Ł = (Ł_x, Ł_y, Ł_z) = \textstyle\sum_{k\ell m}(\tfrac{1}{2}[C^+_{\ell m-1}\, a^{in\dagger}_{k\ell m} a^{in}_{k\ell m-1} + C^-_{\ell m}\, a^{in\dagger}_{k\ell m-1} a^{in}_{k\ell m}], \\ \tfrac{-\imath}{2}[C^+_{\ell m}\, a^{in\dagger}_{k\ell m+1} a^{in}_{k\ell m} - C^-_{\ell m}\, a^{in\dagger}_{k\ell m-1} a^{in}_{k\ell m}], m\, a^{in\dagger}_{k\ell m} a^{in}_{k\ell m}) \end{aligned} \tag{46}$$

with

$$C^+_{\ell m} = \sqrt{(\ell - m)(\ell + m + 1)} \qquad\qquad C^-_{\ell m} = \sqrt{(\ell + m)(\ell - m + 1)} \tag{47}$$

The expression of $Ł$ in terms of the "*in*" creation and annihilation operators is valid for all times. Furthermore, as we work in Heisenberg picture, the state of the electromagnetic field is always the "*in*" vacuum, so we have

$$< Ł >=< 0_{in}|Ł|0_{in} >= 0 \qquad \forall t \tag{48}$$

However, in the "*out*" region, when the radius of the cavity returns to its original value and remains at rest, we could expand the field in terms of the new set of the "*out*" creation and annihilation operators. Therefore the angular momentum of the field can be written as in Eq.(46) but changing the "*in*" operators by the "*out*" ones. As the number of "*out*" particles is different from zero, photons must be created forming singlet states, in such a way that the angular momentum of the field remains null.

The state of the field $|0_{in}>$ can be written as a linear combination of "*out*" states

$$|0_{in} >= \alpha|0_{out} > + \sum_{n\ell m} \alpha_{n\ell m}\, a^{out\dagger}_{n\ell m}|0_{out} > + \sum_{n\ell m}\sum_{n'\ell' m'} \alpha_{n\ell m,n'\ell' m'}\, a^{out\dagger}_{n\ell m} a^{out\dagger}_{n'\ell' m'}|0_{out} > +..... \tag{49}$$

In the parametric case $\Omega = 2\omega_{LN}$, the relation between the "*in*" and "*out*" operators is, for $\ell = L$ and $n = N$,

$$a^{out}_{NLm} = [B^{(N)}_{LN} a^{in}_{NLm} + (-1)^m (A^{(N)}_{LN})^* a^{in\dagger}_{NL-m}]\sqrt{2\omega_{LN}} \tag{50}$$

$$a^{out\dagger}_{NLm} = [(B^{(N)}_{LN})^* a^{in\dagger}_{NLm} + (-1)^m A^{(N)}_{LN} a^{in}_{NL-m}]\sqrt{2\omega_{LN}} \tag{51}$$

The "*in*" and "*out*" operators coincide $a^{out}_{n\ell m} = a^{in}_{n\ell m}$ when $\ell \neq L$ or $n \neq N$. Therefore, we rewrite Eq.(49) in the following way

$$|0_{in} >= \alpha|0_{out} > + \sum_m \alpha_m |1_{out} >_m + \sum_m \sum_{m'} \alpha_{m,m'} |1_{out} >_m |1_{out} >_{m'} +..... \tag{52}$$

where we omitted the subindexes N and L.

In order to find the α coefficients we apply a destruction "*in*" operator on both sides of Eq. (52). The left hand side gives zero. On the right hand side we write the "*in*" operator in terms of the "*out*" operators, by inverting Eqs.(50)-(51). Doing that we obtain a linear combination of orthogonal states which equals zero, so the coefficient of each state must vanish. In this way we get the equations that determine the α coefficients. In the particular case of $\Omega = 2\omega_{LN}$ with L=1, the state of the field $|0_{in}>$ can be written as

$$\begin{gathered}
|0_{in}>= A\{|0_{out}> -\mathcal{C}[|1_{out}>_1 |1_{out}>_{-1} - \frac{1}{\sqrt{2}}|2_{out}>_0] + \\
\mathcal{C}^2[|2_{out}>_1 |2_{out}>_{-1} - \frac{1}{\sqrt{2}}|1_{out}>_1 |2_{out}>_0 |1_{out}>_{-1} + \frac{\sqrt{3}}{2\sqrt{2}}|4_{out}>_0] - \\
\mathcal{C}^3[|3_{out}>_1 |3_{out}>_{-1} - \frac{1}{\sqrt{2}}|2_{out}>_1 |2_{out}>_0 |2_{out}>_{-1} + \\
\frac{\sqrt{3}}{2\sqrt{2}}|1_{out}>_1 |4_{out}>_0 |1_{out}>_{-1} - \frac{\sqrt{5}}{4}|6_{out}>_0] + ...\}
\end{gathered} \tag{53}$$

where A is the normalization constant of the state and

$$\mathcal{C} = -\tanh(\gamma\epsilon t_f)\ . \tag{54}$$

The constant γ is given by Eq.(38) for TE photons and by Eq.(39) for TM photons, both with $L = 1$. Each state between brackets in Eq. (53) is an eigenstate of $\mathbf{L}^2$ and L_z with eigenvalue equal to zero, as expected.

VI. CONCLUSIONS

In this paper we have computed the resonant photon creation inside a spherical oscillating cavity taking into account the vector nature of the electromagnetic field. We described the TE and TM modes of the electromagnetic field by massless scalar fields satisfying Dirichlet and generalized Neumann boundary conditions. We first studied the creation of particles for these scalar fields, and then showed that the number of created photons in each mode (TE or TM) equals the number of created particles of the corresponding scalar field. We used MSA to take into account resonant effects at long times, and found an exponential growth in the number of created photons. Previous works studied the case of a Dirichlet scalar field only in the short time limit $\epsilon\Omega t << 1$ [9]. Our results are consistent with those in this limit.

The spectrum of the scalar fields is equidistant for $\ell = 0$ and non equidistant for $\ell \neq 0$. When the external frequency is chosen to produce parametric resonance in modes with $\ell = 0$, an infinite number of modes are excited. The problem becomes equivalent to the one dimensional dynamical Casimir effect. The modes with $\ell \neq 0$ are non equidistant, and there is no mode coupling. In the parametric resonance case $\Omega = 2\omega_{\ell n}$, the number of motion-induced particles grows exponentially in the particular modes corresponding to these values of n and ℓ, for all possible values of m.

The eigenfrequencies of the TE and TM modes of the electromagnetic field are the same as the eigenfrequencies of ϕ^{TE} and ϕ^{TM} (the only difference is that in the electromagnetic case the $\ell = 0$ modes are absent). The growth rate is different for TE and TM modes. In cubic cavities, where the frequencies of the TE and TM modes are equal, they satisfy [10]

$$\gamma^{TE} < \frac{\omega}{2} \qquad \gamma^{TM} > \frac{\omega}{2} \tag{55}$$

For spherical cavities, the TE and TM modes have different eigenfrequencies. The growth rates satisfy

$$\gamma^{TE} = \frac{\omega^{TE}}{2} \qquad \gamma^{TM} > \frac{\omega^{TM}}{2} \tag{56}$$

Therefore the ratio between the rate of growth and the eigenfrequency is, as in cubic cavities, larger for TM modes than for TE modes. In other words, the functional dependence of the rate of growth with the eigenfrequency is different for TE and TM modes.

Finally, we considered the conservation of the angular momentum of the electromagnetic field in the oscillating cavity. As the initial state of the field is the vacuum, and the motion of the shell does not break the symmetry of rotation, the angular momentum of the field must be zero for all time. Therefore, particles must be created in singlet states. For the particular case $\Omega = 2\omega_{1N}$ we wrote explicitly the "*in*" vacuum state as a linear combination of "*out*" singlet states.

In classical electromagnetism there is no electromagnetic radiation with spherical symmetry. Therefore it was not completely obvious that the dynamical Casimir effect would occur in a spherical oscillating cavity. A mechanical analogue of the one dimensional dynamical Casimir effect is useful to illustrate this point [20]. At the classical level it is not possible to amplify transversal oscillations on a string by changing its length, unless there is an initial classical wave on it [20, 21]. At the quantum level, the initial conditions of the modes are non trivial due to Heisenberg uncertainty principle, and therefore it is possible to excite the system even starting from the ground state. Analogously, in the spherically symmetric case, the dynamical Casimir effect is non trivial because all modes with $\ell \neq 0$ have non vanishing quantum fluctuations.

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VIII. APPENDIX

In this Appendix we solve Eqs.(17) for an oscillating motion of the shell given by $a(t) = a_0(1 + \epsilon \sin(\Omega t))$. For small amplitudes ($\epsilon \ll 1$), Eqs. (17) take the form

$$\begin{aligned}\ddot{Q}^{(k)}_{n,TE}(t) + [\omega_{\ell n}]^2 Q^{(k)}_{n,TE}(t) = 2\epsilon \sin(\Omega t)[\omega_{\ell n}]^2 Q^{(k)}_{n,TE}(t)\\ +2\epsilon\Omega\cos(\Omega t)\textstyle\sum_p \dot{Q}^{(k)}_{p,TE}(t) g^\ell_{pn} - \epsilon\Omega^2 \sin(\Omega t)\sum_p Q^{(k)}_{p,TE}(t) g^\ell_{pn}\\ +\epsilon^2\Omega^2\cos^2(\Omega t)\textstyle\sum_{pN} Q^{(k)}_{p,TE}(t) g^\ell_{pN} g^\ell_{nN} + \mathcal{O}(\epsilon^2)\end{aligned} \tag{57}$$

It is well known that a naive perturbative solution of these equations in powers of ϵ breaks down after a short amount of time, of order $(\epsilon\Omega)^{-1}$. This happens for those particular values of the external frequency Ω such that there is a resonant coupling with the eigenfrequencies of the static cavity. In this situation, to find a solution valid for longer times (of order $(\epsilon^{-2}\Omega^{-1})$) we use the MSA technique [7, 18]. We introduce a second time scale $\tau = \epsilon t$ and expand $Q^{(k)}_{n,TE}(t)$ as follows

$$Q^{(k)}_{n,TE} = Q^{(k,0)}_{n,TE}(t,\tau) + \epsilon Q^{(k,1)}_{n,TE}(t,\tau) + \mathcal{O}(\epsilon^2) \tag{58}$$

The initial conditions read

$$Q^{(k,0)}_{n,TE}(t=0) = \frac{\delta_{kn}}{\sqrt{2\omega_{\ell k}}} \qquad\qquad \dot{Q}^{(k,0)}_{n,TE}(t=0) = -\imath\sqrt{\frac{\omega_{\ell k}}{2}}\delta_{kn} \tag{59}$$

Replacing Eq.(58) into Eq.(57), to zeroth order in ϵ we get the equation of an harmonic oscillator. The solution is

$$Q^{(k,0)}_{n,TE}(t,\tau) = A^{(k)}_{\ell n,TE}(\tau)e^{\imath\omega_{\ell n}t} + B^{(k)}_{\ell n,TE}(\tau)e^{-\imath\omega_{\ell n}t} \tag{60}$$

and using the initial conditions it follows that

$$A^{(k)}_{\ell n,TE}(\tau=0) = 0 \qquad\qquad B^{(k)}_{\ell n,TE}(\tau=0) = \frac{1}{\sqrt{2\omega_{\ell k}}}\delta_{kn} \tag{61}$$

To first order in ϵ we obtain

$$\partial^2_{tt} Q^{(k,1)}_{n,TE}(t,\tau) + [\omega_{\ell n}]^2 Q^{(k,1)}_{n,TE}(t,\tau) = -2\partial_{\tau t} Q^{(k,0)}_{n,TE}(t,\tau) + 2\sin(\Omega t)[\omega_{\ell n}]^2 Q^{(k,0)}_{n,TE}(t,\tau)$$
$$+2\Omega\cos(\Omega t)\sum_p \partial_t Q^{(k,0)}_{p,TE}(t,\tau) g^{\ell}_{pn} - \Omega^2 \sin(\Omega t)\sum_p Q^{(k,0)}_{p,TE}(t,\tau) g^{\ell}_{pn} \tag{62}$$

The functions $A^{(k)}_{\ell n,TE}(\tau)$ and $B^{(k)}_{\ell n,TE}(\tau)$ are obtained by imposing that no secular terms appear in the equation for $Q^{(k,1)}_{n,TE}(t,\tau)$, that is, any term with a time dependence of the form $e^{\pm i\omega_{\ell k}t}$ in the right-hand side of Eq.(62) must vanish. We get

$$\partial_\tau A^{(k)}_{\ell n,TE} = -\tfrac{\omega_{\ell n}}{2} B^{(k)}_{\ell n,TE}(\tau)\delta(\Omega - 2\omega_{\ell n}) + \textstyle\sum_{p\neq n}(\tfrac{\Omega}{2} - \omega_{\ell p}) B^{(k)}_{\ell p,TE}(\tau) g^{\ell}_{pn}\tfrac{\Omega}{2\omega_{\ell n}}$$
$$\delta(\Omega - \omega_{\ell n} - \omega_{\ell p}) + \textstyle\sum_{p\neq n}[(\omega_{\ell p} + \tfrac{\Omega}{2})\delta(\Omega + \omega_{\ell p} - \omega_{\ell n}) +$$
$$(\omega_{\ell p} - \tfrac{\Omega}{2})\delta(\Omega + \omega_{\ell n} - \omega_{\ell p})] A^{(k)}_{\ell p,TE}(\tau) g^{\ell}_{pn}\tfrac{\Omega}{2\omega_{\ell n}} \tag{63}$$

$$\partial_\tau B^{(k)}_{\ell n,TE} = -\tfrac{\omega_{\ell n}}{2} A^{(k)}_{\ell n,TE}\delta(\Omega - 2\omega_{\ell n}) + \textstyle\sum_{p\neq n}(\tfrac{\Omega}{2} - \omega_{\ell p}) A^{(k)}_{\ell p,TE} g^{\ell}_{pn}\tfrac{\Omega}{2\omega_{\ell n}}$$
$$\delta(\Omega - \omega_{\ell n} - \omega_{\ell p}) + \textstyle\sum_{p\neq n}[(\omega_{\ell p} + \tfrac{\Omega}{2})\delta(\Omega + \omega_{\ell p} - \omega_{\ell n}) +$$
$$(\omega_{\ell p} - \tfrac{\Omega}{2})\delta(\Omega + \omega_{\ell n} - \omega_{\ell p})] B^{(k)}_{\ell p,TE} g^{\ell}_{pn}\tfrac{\Omega}{2\omega_{\ell n}} \tag{64}$$

The equations above lead to resonant behavior if

$$\Omega = 2\omega_{\ell n} \qquad \text{or} \qquad \Omega = \omega_{\ell n} + \omega_{\ell p} \tag{65}$$

These are the resonant conditions. The modes (ℓ, n) and (ℓ, q) are coupled if any of the following conditions is satisfied

$$\Omega = \omega_{\ell n} - \omega_{\ell q} \qquad \Omega = -\omega_{\ell n} + \omega_{\ell q} \tag{66}$$

These are the intermode coupling conditions.

The eigenfrequencies are determined by the zeros of the spherical Bessel functions ($\omega_{\ell n} = \frac{j_{\ell n}}{a_0}$). For $\ell = 0$ the spectrum is equidistant and therefore the intermode coupling conditions are satisfied. However, as for the electromagnetic field there are no modes with $\ell = 0$, in what follows we will consider only the case $\ell \neq 0$, in which the spectrum is not equidistant [19], and one can check that the intermode coupling conditions are not satisfied.

For the particular case $\Omega = 2\omega_{LN}$, Eqs.(63,64) become

$$\partial_\tau A^{(k)}_{\ell n,TE} = -\frac{\omega_{\ell n}}{2} B^{(k)}_{\ell n,TE}\delta_{\ell L}\delta_{nN} \tag{67}$$

$$\partial_\tau B^{(k)}_{\ell n,TE} = -\frac{\omega_{\ell n}}{2} A^{(k)}_{\ell n,TE}\delta_{\ell L}\delta_{nN} \tag{68}$$

Using the initial conditions (61) we obtain, for $\ell = L$ and $n = N$

$$A^{(k)}_{LN,TE} = \frac{-\delta_{kN}}{\sqrt{2\omega_{LN}}}\sinh(\frac{\omega_{LN}}{2}\tau) \tag{69}$$

$$B^{(k)}_{LN,TE} = \frac{\delta_{kN}}{\sqrt{2\omega_{LN}}}\cosh(\frac{\omega_{LN}}{2}\tau) \tag{70}$$

and for $\ell \neq L$

$$A^{(k)}_{\ell n,TE} = 0 \qquad\qquad B^{(k)}_{\ell n,TE} = \frac{\delta_{kn}}{\sqrt{2\omega_{\ell k}}} \tag{71}$$

If the motion of the shell ends at $t = t_f$, the number of created particles is given by

$$< \mathcal{N}_{NLm} >=< 0_{in}|a^{out\dagger}_{NLm} a^{out}_{NLm}|0_{in} >= \sinh^2(\frac{\omega_{LN}}{2}\epsilon t_f) \tag{72}$$

$$< \mathcal{N}_{NL} >= \sum_m < 0_{in}|a^{out\dagger}_{NLm} a^{out}_{NLm}|0_{in} >= (2L+1)\sinh^2(\frac{\omega_{LN}}{2}\epsilon t_f) \tag{73}$$

In this Appendix we only considered the case of the scalar field ϕ^{TE}, associated to the TE modes. The Eqs.(27) for the scalar field ϕ^{TM} can also be solved using MSA, following the procedure described here.

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